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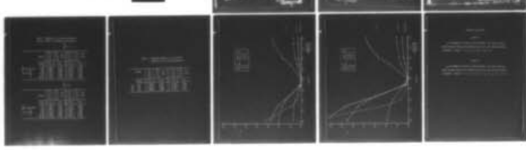

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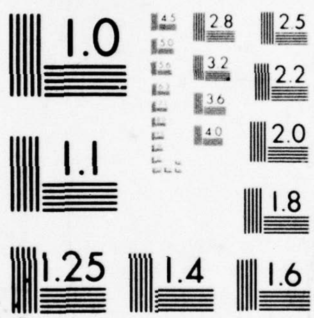

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Sensitivity of MINQUE with
Respect to A Priori Weights

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SUMMARY

The variances of the MINQU estimators of the variance components for the random effects model $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ depend on the values of a priori weights needed for the estimators. The MINQUE have minimum variance in the class of unbiased, quadratic estimators when (i) α_i and ε_{ij} are normally distributed and (ii) the ratio of the a priori weights is equal to the ratio of the variances of α_i and ε_{ij}. This paper shows that for the a priori weights in a neighborhood of the ratio of the variance components the variances of the MINQUE are quite insensitive. That is, the variances deviate little from the optimum variances.

For comparison, the variances of the corresponding Henderson Method I or analysis of variance type estimators of the variance components are also computed. Recommendations can be made as to when to use which type of estimator and how to specify the a priori weights if a MINQU estimator is used.

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The variances of the MINQU estimators of the variance components for the random effects model $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ depend on the values of a priori weights needed for the estimators. The MINQUE have minimum variance in the class of unbiased, quadratic estimators when (i) α_i and ε_{ij} are normally distributed and (ii) the ratio of the a priori weights is equal to the ratio of the variances of α_i and ε_{ij}.

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20 Abstract

This paper shows that for the a priori weights in a neighborhood of the ratio of the variance components the variances of the MINQUE are quite insensitive. That is, the variances deviate little from the optimum variances.

For comparison, the variances of the corresponding Henderson Method I or analysis of variance type estimators of the variance components are also computed. Recommendations can be made as to when to use which type of estimator and how to specify the a priori weights if a MINQU estimator is used.

I. Notation

Consider the one way random effect model with ℓ levels of the random factor and r_i observations in the ith level, $i = 1, 2, \ldots, \ell$. The model can be written as

$$\underset{n\times 1}{\underline{Y}} = \mu \underline{1}_n + U_1 \underline{\alpha} + \underline{\varepsilon}$$

where $n = \sum_{i=1}^{\ell} r_i$, μ is an unknown mean, $\underline{1}_n$ is an $n \times 1$ vector of ones,

$$\underset{n\times \ell}{U_1} = \begin{bmatrix} \underline{1}_{r_1} & & & \phi \\ & \underline{1}_{r_2} & & \\ & & \ddots & \\ \phi & & & \underline{1}_{r_\ell} \end{bmatrix}$$

is the design matrix, $\underline{\alpha} \sim N(\underline{0}, \sigma_a^2 I)$, $\underline{\varepsilon} \sim N(\underline{0}, \sigma_e^2 I)$, and the elements of $\underline{\alpha}$ and $\underline{\varepsilon}$ are mutually independent. Note that $\underline{Y} \sim N(\mu \underline{1}_n, V)$ where $V = \sigma_a^2 V_1 + \sigma_e^2 I$ and $V_1 = U_1 U_1'$.

The minimum norm quadratic unbiased estimator (MINQUE), denoted

$$\hat{\underline{\sigma}} = \begin{bmatrix} \hat{\sigma}_a^2 \\ \hat{\sigma}_e^2 \end{bmatrix},$$

of σ_a^2 and σ_e^2 is given as the solution of $S\hat{\underline{\sigma}} = \underline{u}$ (Rao [1972]) where the matrix S and the vector $\underline{u}$ will be defined below. Both S and $\underline{u}$ depend on the matrix

$$R = \left[V^{*-1} - \frac{V^{*-1} J_n V^{*-1}}{(\underline{1}' V^{*-1} \underline{1})} \right]$$

where $V^* = w_a V_1 + w_e I$, J_n is an $(n \times n)$ matrix of ones and w_a and w_e are a priori weights which are needed for the estimator (actually, the estimator depends on the a priori weights only through the ratio $\frac{w_a}{w_e}$).

The problem considered here is that when $\underline{\alpha}$ and $\underline{\varepsilon}$ are normally distributed the MINQU estimators have minimum variance in the class of unbiased quadratic estimators only when the ratio of the a priori weights equals the ratio of the variance components, i.e. $\frac{w_a}{w_e} = \frac{\sigma_a^2}{\sigma_e^2}$. At first glance, it would seem that this is a very severe requirement to assure a desired optimal property of the unbiased quadratic estimator; however, it was found that the variances of $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ remained close to the minimum variances for $\frac{w_a}{w_e}$ in a large neighborhood of $\frac{\sigma_a^2}{\sigma_e^2}$ for several values of the parameters and several degrees of unbalancedness of the design. This is discussed in detail in section III.

For comparison, the variances of the Henderson Method I estimators of σ_e^2 and σ_a^2 were computed. These quadratic, unbiased estimators are easily calculated and it is of interest to note the magnitude of their variances relative to the minimum variances attainable in the class of unbiased, quadratic estimators as well as the variances of the MINQU estimators which depend on the ratio $\frac{w_a}{w_e}$.

The Henderson Method I estimators of σ_a^2 and σ_e^2 are given as (Searle [1968])

$$\tilde{\sigma}_a^2 = \frac{n[(n-\ell)\cdot SSA - (\ell-1)\cdot SSE]}{(n-\ell)(n^2 - \sum_{i=1}^{\ell} r_i^2)}$$

and

$$\tilde{\sigma}_e^2 = \frac{SSE}{(n-\ell)}$$

respectively where SSA and SSE will be defined below.

It is important to note that SSA and SSE, and hence the variances of $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$, do not depend on a priori weights as do the MINQU estimators.

II. Variances

For the random effects model described above, the S matrix, $\underline{u}$ vector, SSA, and SSE are defined as follows:

$$\underset{2\times 2}{S} = [S_{ij}]$$

where $S_{11} = tr(RV_1RV_1)$, $S_{12} = S_{21} = tr(RV_1R)$, and $S_{22} = tr(RR)$,

$$\underset{2\times 1}{\underline{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $u_1 = \underline{Y}'RV_1R\underline{Y}$ and $u_2 = \underline{Y}'RR\underline{Y}$,

$$SSA = \underline{Y}'[U_1(U_1'U_1)^{-1}U_1' - \frac{1}{n}J_n]\underline{Y} = \underline{Y}'Q_a\underline{Y},$$

and

$$SSE = \underline{Y}'[I-U_1(U_1'U_1)^{-1}U_1']\underline{Y} = \underline{Y}'Q_e\underline{Y}.$$

The MINQU estimators of σ_a^2 and σ_e^2 are then given by

$$\hat{\sigma}_a^2 = \frac{S_{12}u_2 - S_{22}u_1}{S_{12}^2 - S_{11}S_{22}}$$

and

$$\hat{\sigma}_e^2 = \frac{S_{12}u_1 - S_{11}u_2}{S_{12}^2 - S_{11}S_{22}}$$

respectively.

The variances of $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ can now be found as follows:

$$var(\hat{\sigma}_a^2) = k^2\{S_{12}^2 var(u_2) + S_{22}^2 var(u_1) - 2S_{22}S_{12}cov(u_1,u_2)\}$$

and

$$var(\hat{\sigma}_e^2) = k^2\{S_{12}^2 var(u_1) + S_{11}^2 var(u_2) - 2S_{12}S_{11}cov(u_1,u_2)\}$$

where $k = (S_{12}^2 - S_{11}S_{22})^{-1}$. Using results as found in Searle ([1971] pg. 55) and noting that $\underline{1}'R = \underline{0}'$, the following quantities can be obtained:

$$\mathrm{var}(u_1) = 2\mathrm{tr}(RV_1RVRV_1RV),$$

$$\mathrm{var}(u_2) = 2\mathrm{tr}(RRVRRV) \text{ and}$$

$$\mathrm{cov}(u_1,u_2) = 2\mathrm{tr}(RV_1RVRRV)$$

where V is the variance of $\underline{Y}$ as defined above. Hence, the variances of the MINQU estimators can be written

$$\mathrm{var}(\hat{\sigma}_a^2) = 2k^2\{S_{12}^2\mathrm{tr}[(RRV)^2]+S_{22}^2\mathrm{tr}[(RV_1RV)^2]-2S_{22}S_{12}\mathrm{tr}[RV_1RVRRV]\}$$

and

$$\mathrm{var}(\hat{\sigma}_e^2) = 2k^2\{S_{12}^2\mathrm{tr}[(RV_1RV)^2]+S_{11}^2\mathrm{tr}[(RRV)^2]-2S_{12}S_{11}\mathrm{tr}[RV_1RVRRV]\}.$$

The variances of $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$, the Henderson Method I estimators, can be found as above. Noting that the quadratic forms SSA and SSE are independent, the variances of $\tilde{\sigma}_a^2$ and $\tilde{\sigma}_e^2$ can be written as follows:

$$\mathrm{var}(\tilde{\sigma}_a^2) = \frac{n^2}{(n-\ell)^2(n^2-\sum_{i=1}^{\ell} r_i^2)^2}\{(n-\ell)^2\mathrm{var}(SSA)+(\ell-1)^2\mathrm{var}(SSE)\}$$

and

$$\mathrm{var}(\tilde{\sigma}_e^2) = \frac{1}{(n-\ell)^2}\{\mathrm{var}(SSE)\}.$$

Again using results as found in Searle [(1971) pg. 55] and noting that $\underline{1}'Q_a = \underline{1}'Q_e = \underline{0}'$, then

$$\mathrm{var}(SSA) = 2\mathrm{tr}(Q_aVQ_aV)$$

and

$$\mathrm{var}(SSE) = 2\mathrm{tr}(Q_eVQ_eV).$$

Therefore, the variances of the Henderson Method I estimators can be written

$$\operatorname{var}(\tilde{\sigma}_a^2) = \frac{2n^2}{(n-\ell)^2(n^2-\sum_{i=1}^{\ell} r_i^2)^2}\{(n-\ell)^2 \operatorname{tr}(Q_a V Q_a V)+(\ell-1)^2 \operatorname{tr}(Q_e V Q_e V)\}$$

and

$$\operatorname{var}(\tilde{\sigma}_e^2) = \frac{2}{(n-\ell)^2}\{\operatorname{tr}(Q_e V Q_e V)\}.$$

III. Results

The measure of sensitivity of the variance of the estimators is defined as follows:

$$\gamma_a = \frac{\text{variance of the estimator of } \sigma_a^2 \text{ being considered}}{\text{minimum variance in class of quadratic, unbiased estimators}}$$

The measure used does not involve the variance of the estimator of σ_e^2 being considered since for all cases investigated the variance of $\hat{\sigma}_e^2$ remained stable relative to the variance of $\hat{\sigma}_a^2$ for various parameters, a priori weights, and designs. Note that $\gamma_a \geq 1$ and that the decimal portion of γ_a times 100 is the percent by which the variance of the estimator of σ_a^2 being considered exceeds the minimum variance for a quadratic, unbiased estimator. Also, γ_a depends on the variance parameters only through the ratio $\frac{\sigma_a^2}{\sigma_e^2}$.

In order to compare results obtained across different sample sizes and different number of levels of the random effect, a measure of unbalancedness is defined. Using an extension of a measure used by Low [1976],

$$\%\max D = \frac{D}{D_{max}}$$

where $D = \sum_{i=1}^{\ell}(r_i-\bar{r})^2 - \min\left[\sum_{i=1}^{\ell}(r_i-\bar{r})^2\right]$, $\bar{r} = \frac{1}{\ell}\sum_{i=1}^{\ell} r_i = \frac{n}{\ell}$, and D_{max} is the maximum value D can attain. Designs are then chosen for comparison that have approximately the same % max D. Note that when the design is balanced as possible then % max D = 0 and that when the design is as unbalanced as possible then % max D = 1.

The study took many design configurations into consideration. That is, the behavior of γ_a for values of $\frac{\sigma_a^2}{\sigma_e^2}$, $\frac{w_a}{w_e}$, n, ℓ, and r_i $(i=1,2,\ldots\ell)$ was studied. Designs were considered where more degrees of freedom were available to estimate σ_a^2 than σ_e^2; that is, the majority of the $r_i = 1$. The results reported here, however, are only for the case when there are more degrees of freedom to estimate σ_e^2 than σ_a^2. Attention is devoted to this type of design for two reasons. First, this seems to be the case in the majority of physical situations. Second, there is a lack of consistency and trends when more degrees of freedom are available to estimate the random effect variance than the error variance. The results which are tabulated and graphed below are representative of the many design configurations considered.

[Insert Tables and Figures]

IV. Conclusions

When considering designs for which there are more degrees of freedom to estimate σ_e^2 than σ_a^2 and for which the unbalancedness is not extreme, recommendations can be made. As can be seen from tables 1 and 2 the MINQU estimator becomes increasingly sensitive to the a priori weights as the degree of unbalancedness increases. Also, tables 1 and 2 as well as figures 1 and 2 illustrate that as the number of levels of the random factor, ℓ, increases the more sensitive the MINWU estimator becomes to the ratio of the weights, $\frac{w_a}{w_e}$. Again looking at tables 1 and 2, graphs 1 and 2, and excepting the case when the variance ratio is smallest it is seen that the MINQU estimator is more sensitive to the a priori weights which underestimate $\frac{\sigma_a^2}{\sigma_e^2}$ than it is to the a priori weights which in ratio are greater than $\frac{\sigma_a^2}{\sigma_e^2}$.

Incorporating the results obtained for the Henderson analysis of variance type estimator (Table 3) with the results noted above for the MINQUE the following recommendations can be made for the one way random effect model. First, if the experimenter believes that the ratio of variances, $\frac{\sigma_a^2}{\sigma_e^2}$, is less than one then the Henderson Method I estimator should be used. Second, if the experimenter believes that the random effect variance is greater than the error variance then the MINQU estimator should be used incorporating a priori weights which in ratio you feel does not underestimate $\frac{\sigma_a^2}{\sigma_e^2}$.

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Table 1. MINQUE-γ_a for various parameters, a priori weights, and design configurations

$$\frac{\sigma_a^2}{\sigma_e^2} = 0.2$$

	design	$\ell=3$, $r_1=4$, $r_2=6$, $r_3=8$	$\ell=5$, $r_1=2$, $r_2=3$, $r_3=4$, $r_4=5$, $r_5=6$	$\ell=3$, $r_1=2$, $r_2=3$, $r_3=13$	$\ell=5$, $r_1=2$, $r_2=2$, $r_3=2$, $r_4=2$, $r_5=12$
	% max D	0.05	0.06	0.49	0.44
$\frac{w_a}{w_e}$ is this portion of $\frac{\sigma_a^2}{\sigma_e^2}$	0.01	1.046	1.070	1.077	1.187
	0.10	1.032	1.051	1.059	1.142
	0.50	1.005	1.010	1.013	1.030
	2.00	1.005	1.013	1.020	1.039
	5.00	1.018	1.065	1.101	1.171
	10.00	1.027	1.109	1.168	1.263
	100.00	1.038	1.176	1.266	1.378

$$\frac{\sigma_a^2}{\sigma_e^2} = 0.6$$

	design	$\ell=3$, $r_1=4$, $r_2=6$, $r_3=8$	$\ell=5$, $r_1=2$, $r_2=3$, $r_3=4$, $r_4=5$, $r_5=6$	$\ell=3$, $r_1=2$, $r_2=3$, $r_3=13$	$\ell=5$, $r_1=2$, $r_2=2$, $r_3=2$, $r_4=2$, $r_5=12$
	% max D	0.05	0.06	0.49	0.44
$\frac{w_a}{w_e}$ is this portion of $\frac{\sigma_a^2}{\sigma_e^2}$	0.01	1.094	1.181	1.212	1.557
	0.10	1.047	1.107	1.138	1.346
	0.50	1.004	1.012	1.018	1.038
	2.00	1.002	1.008	1.012	1.021
	5.00	1.005	1.028	1.042	1.063
	10.00	1.007	1.040	1.058	1.084
	100.00	1.009	1.053	1.077	1.107

Table 2. MINQUE--γ_a for various parameters, a priori weights, and design configurations

$$\frac{\sigma_a^2}{\sigma_e^2} = 1.0$$

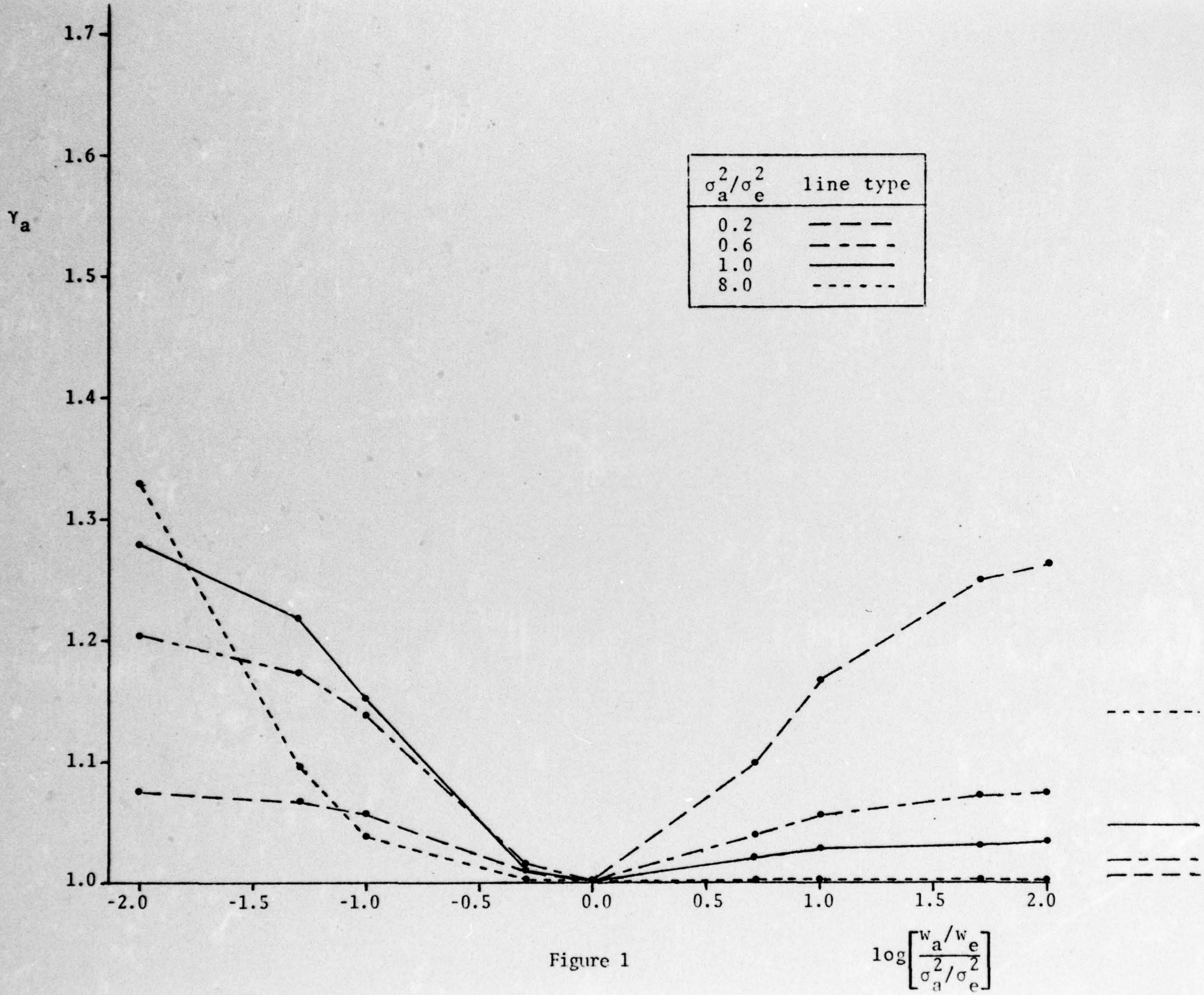
	design	$\ell=3$, $r_1=4$, $r_2=6$, $r_3=8$	$\ell=5$, $r_1=2$, $r_2=3$, $r_3=4$, $r_4=5$, $r_5=6$	$\ell=3$, $r_1=2$, $r_2=3$, $r_3=13$	$\ell=5$, $r_1=2$, $r_2=2$, $r_3=2$, $r_4=2$, $r_5=12$
	% max D	0.05	0.06	0.49	0.44
$\frac{w_a}{w_e}$ is this portion of $\frac{\sigma_a^2}{\sigma_e^2}$	0.01	1.109	1.233	1.281	1.762
	0.10	1.042	1.114	1.157	1.390
	0.50	1.002	1.009	1.014	1.027
	2.00	1.001	1.005	1.007	1.011
	5.00	1.002	1.014	1.021	1.031
	10.00	1.003	1.020	1.028	1.040
	100.00	1.004	1.025	1.036	1.050

$$\frac{\sigma_a^2}{\sigma_e^2} = 8.0$$

	design	$\ell=3$, $r_1=4$, $r_2=6$, $r_3=8$	$\ell=5$, $r_1=2$, $r_2=3$, $r_3=4$, $r_4=5$, $r_5=6$	$\ell=3$, $r_1=2$, $r_2=3$, $r_3=13$	$\ell=5$, $r_1=2$, $r_2=2$, $r_3=2$, $r_4=2$, $r_5=12$
	% max D	0.05	0.06	0.49	0.44
$\frac{w_a}{w_e}$ is this portion of $\frac{\sigma_a^2}{\sigma_e^2}$	0.01	1.078	1.237	1.331	1.895
	0.10	1.004	1.025	1.038	1.069
	0.50	1.000	1.001	1.001	1.001
	2.00	1.000	1.000	1.000	1.000
	5.00	1.000	1.000	1.001	1.001
	10.00	1.000	1.001	1.001	1.001
	100.00	1.000	1.001	1.001	1.001

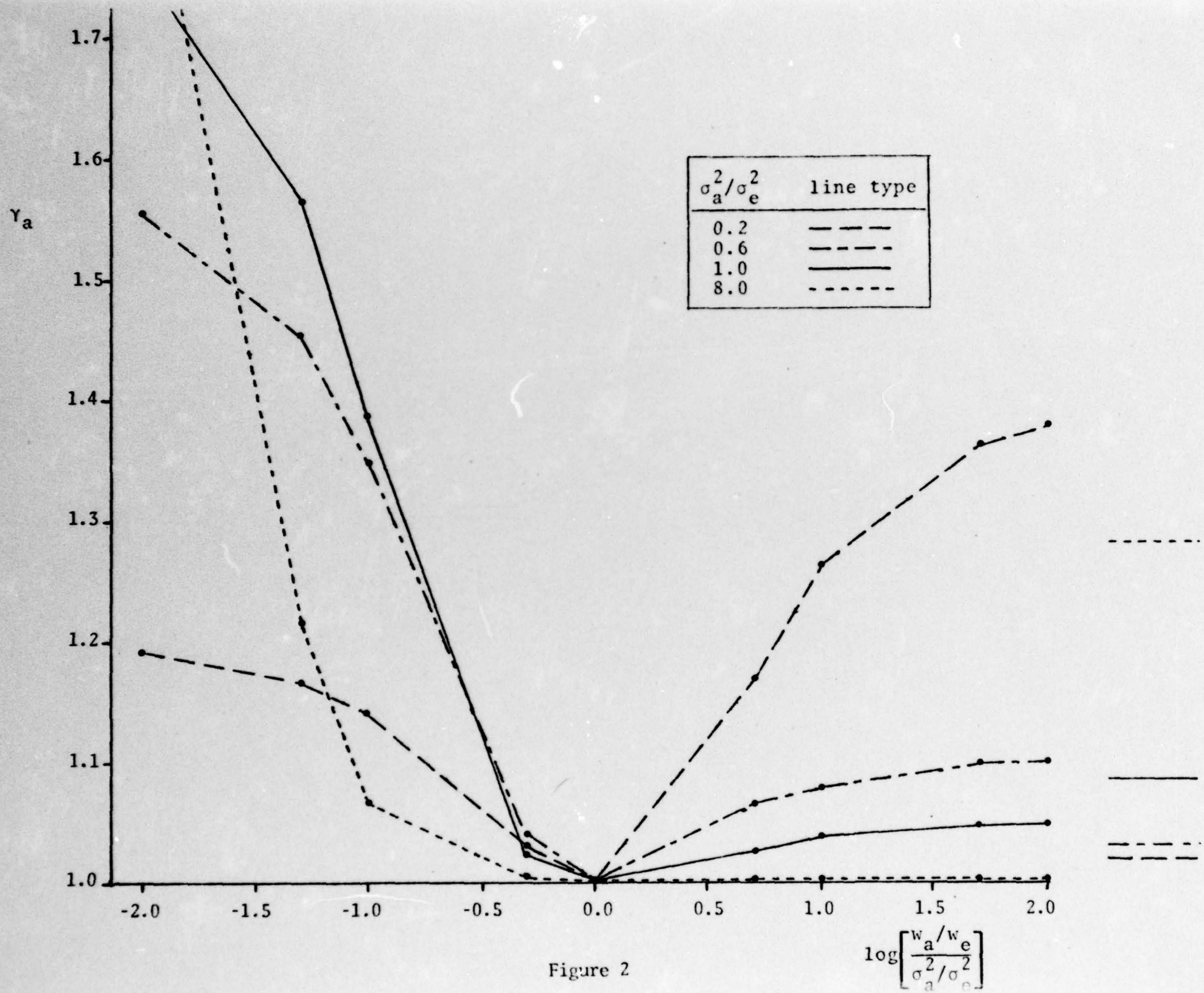
Table 3. Henderson Method I--γ_a for various parameters and design configurations

	design	$\ell=3$ $r_1=4$ $r_2=6$ $r_3=8$	$\ell=5$ $r_1=2$ $r_4=5$ $r_2=3$ $r_5=6$ $r_3=4$	$\ell=3$ $r_1=2$ $r_2=3$ $r_3=13$	$\ell=5$ $r_1=2$ $r_4=2$ $r_2=2$ $r_5=12$ $r_3=2$
	% max D	0.05	0.06	0.49	0.44
$\frac{\sigma_a^2}{\sigma_e^2}$	0.2	1.000	1.006	1.006	1.020
	0.6	1.012	1.011	1.019	1.029
	1.0	1.020	1.030	1.048	1.085
	8.0	1.038	1.089	1.141	1.282



γ_a
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
σ_a^2/σ_e^2
line type
0.2
0.6
1.0
8.0
$\log\left[\frac{w_a/w_e}{\sigma_a^2/\sigma_e^2}\right]$

Figure 1



γ_a
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
σ_a^2/σ_e^2
line type
0.2
0.6
1.0
8.0
$\log\left[\frac{w_a/w_e}{\sigma_a^2/\sigma_e^2}\right]$

Figure 2

Captions for Figures

Figure 1

γ_a for MINQUE for various a priori weights. The short lines on right ordinate indicate the constant value of γ_a for Henderson Method I estimator. Design: $\ell = 3$, $r_1 = 2$, $r_2 = 3$, $r_3 = 13$.

Figure 2

γ_a for MINQUE for various a priori weights. The short lines on right ordinate indicate the constant value of γ_a for Henderson Method I estimator. Design: $\ell = 5$, $r_1 = 2$, $r_2 = 2$, $r_3 = 2$, $r_4 = 2$, $r_5 = 12$.